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# Note on the spaces of real resultants with bounded multiplicity (Geometry, Algebra and Combinatorics in Transformation group theory)

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# Note on the spaces of real resultants with bounded multiplicity

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## Abstract

For positive integers  $d, m, n \geq 1$  with  $(m, n) \neq (1, 1)$  and  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , let  $Q_n^{d,m}(\mathbb{K})$  denote the space of  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in \mathbb{K}[z]^m$  of  $\mathbb{K}$ -coefficients monic polynomials of the same degree  $d$  such that polynomials  $\{f_k(z)\}_{k=1}^m$  have no common *real* root of multiplicity  $\geq n$  (but may have complex common root of any multiplicity). These spaces can be regarded as one of generalizations of the spaces defined and studied by Arnold and Vassiliev [16]. In this paper, we study the homotopy types of  $Q_n^{d,m}(\mathbb{K})$  and announce the results obtained in [12].

## 1 Introduction

**The basic notations.** For connected spaces  $X$  and  $Y$ , let  $\text{Map}(X, Y)$  (resp.  $\text{Map}^*(X, Y)$ ) denote the space consisting of all continuous maps (resp. base point preserving continuous maps) from  $X$  to  $Y$  with the compact-open topology, and let  $\mathbb{RP}^N$  (resp.  $\mathbb{CP}^N$ ) denote the  $N$ -dimensional real projective (resp. complex projective) space. Note that  $\text{Map}(S^1, \mathbb{RP}^N)$  has two path-components  $\text{Map}_\epsilon(S^1, \mathbb{RP}^N)$  for  $\epsilon \in \{0, 1\}$  when  $N \geq 2$ . It is well-known that any map  $f \in \text{Map}_\epsilon(S^1, \mathbb{RP}^N)$  lifts to the map  $F \in \text{Map}(S^1, S^N)$  such that  $F(-x) = (-1)^\epsilon F(x)$  for any  $x \in S^1$ . For each  $\epsilon \in \{0, 1\}$ , let  $\Omega_\epsilon \mathbb{RP}^N$  denote the path component given by  $\Omega_\epsilon \mathbb{RP}^N = \text{Map}_\epsilon(S^1, \mathbb{RP}^N) \cap \text{Map}^*(S^1, \mathbb{RP}^N)$ .

**The motivation.** The principal motivation of this research derived from the results obtained by Vassiliev [16]. For  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , let  $P_n^d(\mathbb{K})$  denote the space of all  $\mathbb{K}$ -coefficients monic polynomials  $f(z) \in \mathbb{K}[z]$  of degree  $d$  which have no *real* root of multiplicity  $\geq n$  (but may have complex ones of arbitrary multiplicity). By identifying  $S^1 = \mathbb{R} \cup \{\infty\}$  and  $\mathbb{C} = \mathbb{R}^2$ , we have the *jet map*

$$(1.1) \quad j_{n,\mathbb{K}}^{d,1} : P_n^d(\mathbb{K}) \rightarrow \Omega_{[d]_2} \mathbb{RP}^{d(\mathbb{K})n-1} \simeq \Omega S^{d(\mathbb{K})n-1}$$

defined by

$$j_{n,\mathbb{K}}^{d,1}(f(z))(\alpha) = \begin{cases} [f(\alpha) : f(\alpha) + f'(\alpha) : \dots : f(\alpha) + f^{(n-1)}(\alpha)] & \text{if } \alpha \in \mathbb{R} \\ [1 : 1 : 1 : \dots : 1] & \text{if } \alpha = \infty \end{cases}$$

for  $(f(z), \alpha) \in P_n^d(\mathbb{K}) \times S^1$ , where  $[d]_2 \in \{0, 1\}$  and  $d(\mathbb{K})$  denote the integers defined by

$$(1.2) \quad [d]_2 = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{2} \\ 0 & \text{if } d \equiv 0 \pmod{2} \end{cases} \quad \text{and} \quad d(\mathbb{K}) = \dim_{\mathbb{R}} \mathbb{K} = \begin{cases} 1 & \text{if } \mathbb{K} = \mathbb{R} \\ 2 & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

For  $\mathbb{K} = \mathbb{R}$ , Vassiliev obtained the following result:

**Theorem 1.1** ([16] (cf. [7], [9])). *The jet map  $j_{n,\mathbb{R}}^{d,1} : P_n^d(\mathbb{R}) \rightarrow \Omega_{[d]_2} \mathbb{R}P^{n-1} \simeq \Omega S^{n-1}$  is a homotopy equivalence through dimension  $(\lfloor \frac{d}{n} \rfloor + 1)(n-2) - 1$  for  $n \geq 4$  and a homology equivalence through dimension  $\lfloor \frac{d}{3} \rfloor$  for  $n = 3$ , where  $\lfloor x \rfloor$  denotes the integer part of a real number  $x$ .  $\square$*

**Remark 1.2.** Remark that a map  $f : X \rightarrow Y$  is called a *homotopy equivalence* (resp. a *homology equivalence*) through dimension  $N$  if the induced homomorphism

$$f_* : \pi_k(X) \rightarrow \pi_k(Y) \quad (\text{resp. } f_* : H_k(X, \mathbb{Z}) \rightarrow H_k(Y, \mathbb{Z}))$$

is an isomorphism for any integer  $k \leq N$ . Similarly, when  $G$  is a group and  $f : X \rightarrow Y$  is a  $G$ -equivariant map between  $G$ -spaces  $X$  and  $Y$ , the map  $f$  is called a  *$G$ -equivariant homotopy equivalence through dimension  $N$*  (resp. a  *$G$ -equivariant homology equivalence through dimension  $N$* ) if the restriction map  $f^H = f|_{X^H} : X^H \rightarrow Y^H$  is a homotopy equivalence through dimension  $N$  (resp. a homology equivalence through dimension  $N$ ) for any subgroup  $H \subset G$ , where  $W^H$  denote the  $H$ -fixed subspace of a  $G$ -space  $W$  given by  $W^H = \{x \in W : h \cdot x = x \text{ for any } h \in H\}$ .  $\square$

The main purpose of this note is to generalize this result given in [9] for the space  $Q_n^{d,m}(\mathbb{K})$ .

**Basic definitions.** From now on, let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , let  $d, m, n \geq 1$  be positive integers such that  $(m, n) \neq (1, 1)$ , and we always assume that  $z$  is a variable.

**Definition 1.3.** (i) Let  $Q_n^{d,m}(\mathbb{K})$  denote the space of  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in P^d(\mathbb{K})^m$  of  $\mathbb{K}$ -coefficients monic polynomials of the same degree  $d$  such that  $f_1(z), \dots, f_m(z)$  have no common *real* root of multiplicity  $\geq n$  (but they may have a common *complex* root of any multiplicity).

(ii) Let  $(f_1(z), \dots, f_m(z)) \in P^d(\mathbb{K})^m$  be an  $m$ -tuple of monic polynomials of the same degree  $d$ . Then it is easy to see that  $(f_1(z), \dots, f_m(z)) \in Q_n^{d,m}(\mathbb{K})$  iff the derivative polynomials  $\{f_j^{(k)}(z) : 1 \leq j \leq m, 0 \leq k < n\}$  have no common real root. Thus, by identifying  $S^1 = \mathbb{R} \cup \infty$ , one can define the *jet map*

$$(1.3) \quad j_{n,\mathbb{K}}^{d,m} : Q_n^{d,m}(\mathbb{K}) \rightarrow \Omega_{[d]_2} \mathbb{R}P^{d(\mathbb{K})mn-1} \simeq \Omega S^{d(\mathbb{K})mn-1} \quad \text{by}$$

$$(1.4) \quad j_{n,\mathbb{K}}^{d,m}(f_1(z), \dots, f_m(z))(\alpha) = \begin{cases} [f_1(\alpha) : \dots : f_m(\alpha)] & \text{if } \alpha \in \mathbb{R} \\ [1 : 1 : \dots : 1] & \text{if } \alpha = \infty \end{cases}$$

for  $(f_1(z), \dots, f_m(z)) \in Q_n^{d,m}(\mathbb{K})$ , where we identify  $\mathbb{C} = \mathbb{R}^2$  in (1.4) if  $\mathbb{K} = \mathbb{C}$ , and  $\mathbf{f}_k(z)$  ( $k = 1, \dots, m$ ) is the  $n$ -tuple of monic polynomials of the same degree  $d$  defined by

$$(1.5) \quad \mathbf{f}_k(z) = (f_k(z), f_k(z) + f'_k(z), f_k(z) + f''_k(z), \dots, f_k(z) + f_k^{(n-1)}(z)).$$

Note that  $P_n^d(\mathbb{K}) = Q_n^{d,1}(\mathbb{K})$ , and that the map  $j_{n,\mathbb{K}}^{d,m}$  coincides the map  $j_{n,\mathbb{K}}^{d,1}$  given in (1.1) for  $m = 1$ . Similarly, one can define a natural map

$$(1.6) \quad i_{n,\mathbb{K}}^{d,m} : Q_n^{d,m}(\mathbb{K}) \rightarrow Q_1^{d,mn}(\mathbb{K}) \quad \text{by}$$

$$(1.7) \quad i_{n,\mathbb{K}}^{d,m}(f_1(z), \dots, f_m(z)) = (\mathbf{f}_1(z), \dots, \mathbf{f}_m(z)).$$

It is well-known that there is a homotopy equivalence

$$(1.8) \quad \Omega S^{N+1} \simeq S^N \cup e^{2N} \cup e^{3N} \cup \dots \cup e^{kN} \cup e^{(k+1)N} \cup \dots.$$

We will denote the  $kN$ -skeleton of  $\Omega S^{N+1}$  by  $J_k(\Omega S^{N+1})$ , i.e.

$$(1.9) \quad J_k(\Omega S^{N+1}) \simeq S^N \cup e^{2N} \cup e^{3N} \cup \dots \cup e^{(k-1)N} \cup e^{kN}.$$

This space is usually called the  $k$ -stage James filtration of  $\Omega S^{N+1}$ . □

**Previous results.** Let  $D(d; m, n, \mathbb{K})$  denote the positive integer defined by

$$(1.10) \quad \begin{aligned} D(d; m, n, \mathbb{K}) &= (d(\mathbb{K})mn - 2)(\lfloor \frac{d}{n} \rfloor + 1) - 1 \\ &= \begin{cases} (2mn - 2)(\lfloor \frac{d}{n} \rfloor + 1) - 1 & \text{if } \mathbb{K} = \mathbb{C}, \\ (mn - 2)(\lfloor \frac{d}{n} \rfloor + 1) - 1 & \text{if } \mathbb{K} = \mathbb{R}. \end{cases} \end{aligned}$$

Recall the following known results for the case  $m = 1$  or  $n = 1$ .

**Theorem 1.4** ([9], [13], [16], [17]). (i) If  $d(\mathbb{K})m \geq 4$  and  $n = 1$ , the jet map

$$j_{1,\mathbb{K}}^{d,m} : Q_1^{d,m}(\mathbb{K}) \rightarrow \Omega_{[d]_2} \mathbb{R}P^{d(\mathbb{K})m-1} \simeq \Omega S^{d(\mathbb{K})m-1}$$

is a homotopy equivalence through dimension  $D(d; m, 1, \mathbb{K})$ .

(ii) If  $d(\mathbb{K})n \geq 4$  and  $m = 1$ , the jet map

$$j_{n,\mathbb{K}}^{d,1} : Q_n^{d,1}(\mathbb{K}) = P_n^d(\mathbb{K}) \rightarrow \Omega_{[d]_2} \mathbb{R}P^{d(\mathbb{K})n-1} \simeq \Omega S^{d(\mathbb{K})n-1}$$

is a homotopy equivalence through dimension  $D(d; 1, n, \mathbb{K})$ .

(iii) If  $d(\mathbb{K})m \geq 4$  and  $d(\mathbb{K})n \geq 1$ , there are homotopy equivalences

$$Q_n^{d,1}(\mathbb{K}) = P_n^d(\mathbb{K}) \simeq J_{\lfloor \frac{d}{n} \rfloor}(\Omega S^{d(\mathbb{K})n-1}) \quad \text{and} \quad Q_1^{d,m}(\mathbb{K}) \simeq J_d(\Omega S^{d(\mathbb{K})m-1}).$$

Thus, there is a homotopy equivalence  $Q_n^{d,1}(\mathbb{K}) = P_n^d(\mathbb{K}) \simeq Q_1^{\lfloor \frac{d}{n} \rfloor, n}(\mathbb{K})$ .

(iv) In particular, if  $(\mathbb{K}, m) = (\mathbb{R}, 3)$  and  $d \geq 1$  is an odd integer, there is a homotopy equivalence  $Q_1^{d,3}(\mathbb{R}) \simeq J_d(\Omega S^2)$ . □

Note that the conjugation on  $\mathbb{C}$  naturally induces a  $\mathbb{Z}/2$ -action on the space  $Q_n^{d,m}(\mathbb{C})$ . From now on, we regard  $\mathbb{RP}^N$  as the  $\mathbb{Z}/2$ -space with trivial  $\mathbb{Z}/2$ -action, and recall the following result given in [9].

**Theorem 1.5** ([9]). (i) *If  $m \geq 4$ , then the jet map*

$$j_{1,\mathbb{C}}^{d,m} : Q_1^{d,m}(\mathbb{C}) \rightarrow \Omega_{[d]_2} \mathbb{RP}^{2m-1} \simeq \Omega S^{2m-1}$$

*is a  $\mathbb{Z}/2$ -equivariant homotopy equivalence through dimension  $D(d; m, 1, \mathbb{R})$ .*

(ii) *If  $n \geq 4$ , then the jet map*

$$j_{n,\mathbb{C}}^{d,1} : Q_n^{d,1}(\mathbb{C}) = P_n^d(\mathbb{C}) \rightarrow \Omega_{[d]_2} \mathbb{RP}^{2n-1} \simeq \Omega S^{2n-1}$$

*is a  $\mathbb{Z}/2$ -equivariant homotopy equivalence through dimension  $D(d; 1, n, \mathbb{R})$ .* □

## 2 The main results

The main purpose of this paper is to study the homotopy type of the space  $Q_n^{d,m}(\mathbb{K})$  and report about the generalizations of the above two theorems (Theorems 1.4 and 1.5) for the case  $m \geq 2$  and the case  $n \geq 2$ . Note that the following results may be regarded as one of real analogues of the result obtained in [11] (cf. [5]). More precisely, the main results are stated as follows.

**Theorem 2.1.** *If  $d(\mathbb{K})mn \geq 4$ , the jet map*

$$j_{n,\mathbb{K}}^{d,m} : Q_n^{d,m}(\mathbb{K}) \rightarrow \Omega_{[d]_2} \mathbb{RP}^{d(\mathbb{K})mn-1} \simeq \Omega S^{d(\mathbb{K})mn-1}$$

*is a homotopy equivalence through dimension  $D(d; m, n, \mathbb{K})$ .* □

Note that the conjugation on  $\mathbb{C}$  naturally induces the  $\mathbb{Z}/2$ -action on the space  $Q_n^{d,m}(\mathbb{C})$ . Since the map  $j_{n,\mathbb{C}}^{d,m}$  is a  $\mathbb{Z}/2$ -equivariant map and  $(j_{n,\mathbb{C}}^{d,m})^{\mathbb{Z}/2} = j_{n,\mathbb{R}}^{d,m}$ , we also obtain the following result.

**Corollary 2.2.** *If  $mn \geq 4$ , the jet map*

$$j_{n,\mathbb{C}}^{d,m} : Q_n^{d,m}(\mathbb{C}) \rightarrow \Omega_{[d]_2} \mathbb{RP}^{2mn-1} \simeq \Omega S^{2mn-1}$$

*is a  $\mathbb{Z}/2$ -equivariant homotopy equivalence through dimension  $D(d; m, n, \mathbb{R})$ .* □

**Corollary 2.3.** *If  $d(\mathbb{K})mn \geq 4$ , the jet embedding*

$$i_{n,\mathbb{K}}^{d,m} : Q_n^{d,m}(\mathbb{K}) \rightarrow Q_1^{d,mn}(\mathbb{K})$$

*is a homotopy equivalence through dimension  $D(d; m, n, \mathbb{K})$ .* □

**Theorem 2.4.** *If  $d(\mathbb{K})mn \geq 4$ , there is a homotopy equivalence*

$$Q_n^{d,m}(\mathbb{K}) \simeq J_{\lfloor \frac{d}{n} \rfloor}(\Omega S^{d(\mathbb{K})mn-1}).$$

*Thus, there are homotopy equivalences  $Q_n^{d,m}(\mathbb{K}) \simeq Q_{mn}^{d,1}(\mathbb{K}) \simeq Q_1^{\lfloor \frac{d}{n} \rfloor, mn}(\mathbb{K})$ .* □

**Remark 2.5.** (i) The above results can be proved by using the Vassiliev spectral sequence ([1], [14], [16]) and the scanning maps ([6], [7], [8], [15]). The detail of their proofs are omitted and see [12] in detail.

(ii) For positive integers  $d, m, n \geq 1$  with  $(m, n) \neq (1, 1)$  and a field  $\mathbb{F}$  with its algebraic closure  $\overline{\mathbb{F}}$ , let  $\text{Poly}_n^{d,m}(\mathbb{F})$  denote the space of all  $m$ -tuples  $(f_1(z), \dots, f_m(z)) \in \mathbb{F}[z]^m$  of monic  $\mathbb{F}$ -coefficients polynomials of the same degree  $d$  such that polynomials  $\{f_k(z)\}_{k=1}^m$  have no common root in  $\overline{\mathbb{F}}$  of multiplicity  $\geq n$ . The space  $\text{Poly}_n^{d,m}(\mathbb{F})$  is first defined and studied by B. Farb and J. Wolfson [5] for investigation the homological density of algebraic cycles in a closed manifold. By the classical theory of resultants, the space  $\text{Poly}_n^{d,m}(\mathbb{C})$  is an affine variety defined by systems of polynomial equations  $\{F_k\}_{k=1}^N$  with integer coefficients. Thus both varieties given by this system of equations can be defined over  $\mathbb{Z}$  and (by extension of scalars or reduction modulo a prime number) over any field  $\mathbb{F}$ . So  $\text{Poly}_n^{d,m}(\mathbb{F})$  is an affine variety for any field  $\mathbb{F}$ .

(iii) Since this system of equations can be obtained by using the generalized resultants, we shall call the space  $\text{Poly}_n^{d,m}(\mathbb{C})$  as the space of resultants with bounded multiplicity. Note that the space  $Q_n^{d,m}(\mathbb{K})$  can be regarded as one of generalizations of real analogues of the space  $\text{Poly}_n^{d,m}(\mathbb{C})$ . Because of this reason, we shall call the space  $Q_n^{d,m}(\mathbb{K})$  as the space of *real* resultants of bounded multiplicity although it is not an affine variety.

(iv) The homotopy type of the space  $\text{Poly}_n^{d,m}(\mathbb{C})$  was already well investigated in [11] (cf. [3], [4], [7], [15]). □

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